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L-isothermic and L-minimal surfaces

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Abstract
It has recently been shown that a second-order linear nonhomogeneous differential equation is associated with a surface with an isothermic representation of their lines of curvature (L-isothermic surface) (Schief *et al* 2007 *J. Math. Phys.* **48** 073510). The 6-parameter group $SL(2, \mathbb{C})$ acting on linearly independent solutions of the homogeneous version of the latter equation generates a Laguerre transformation of the surface. The Weierstrass representation of the surfaces which are both L-isothermic and L-minimal is presented.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Surfaces with isothermic representation of their lines of curvature (L-isothermic surfaces) appear naturally in the context of Laguerre geometry as a subgeometry of Lie sphere geometry [2]. They are defined by the requirement that the lines of curvature are conformal with respect to the third fundamental form g_{III} of the surface. The explicit study of such surfaces goes back to the work of Blaschke [1]. He also investigated the variational problem for the functional $\mathcal{W} = \int(\mathcal{H}^2/\mathcal{K} - 1)\,\mathrm{d}V$ induced by immersed surfaces. Critical points of $\mathcal{W}$ are called L-minimal surfaces. In recent times E Musso and L Nicolodi used the Cartan method of moving frames to describe surfaces in Laguerre geometry. Their considerations were based on the space $\Lambda = \mathbb{E}^3 \times S_2$ of contact elements of $\mathbb{E}^3$. In a series of interesting articles [6–12] they reveal many features of L-isothermic and L-minimal surfaces (Cauchy problem, deformations, Bianchi–Darboux transform). It should also be remarked that the integrability aspects of Lie sphere geometry have been studied by Ferapontov [5].

On the other hand L-isothermic surfaces occur in mathematical physics. It has recently been shown that classical shell membranes for which the stress resultants are not uniquely determined are necessarily L-isothermic surfaces [17, 19]. They are also incorporated in the system of equilibrium equations in liquid crystal theory [18]. In addition the Laguerre geometry, L-isothermic and L-minimal surfaces have also found applications in computer-aided geometric design (CAGD), computational geometry [14] and even architecture [15].

In sections 2 and 3 we recall the geometry of two-dimensional surfaces embedded in three-dimensional Euclidean space $\mathbb{E}^3$ and the basic construction of L-isothermic surfaces [19]. The Laguerre transformations of the surfaces are discussed in section 4. In section 5, the Weierstrass representation of the surfaces which are both L-isothermic and L-minimal is derived. Two examples are presented in section 6.

2. Geometry of two-dimensional surfaces in Euclidean space

We describe the geometry of a two-dimensional surface $\mathbf{\Sigma}$ embedded in a three-dimensional Euclidean space $\mathbb{E}^3$ using a moving frame. Therefore, there is a right-handed orthonormal frame $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)^T$ attached to each point of $\mathbb{E}^3$. We choose a moving frame $\mathbf{e}$ at each point $\mathbf{r}$ of $\mathbf{\Sigma}$ such that $\mathbf{e}_3 = \mathbf{N}$ is the normal to $\mathbf{\Sigma}$. Then $\mathbf{e}_1$ and $\mathbf{e}_2$ are two tangent vectors to $\mathbf{\Sigma}$ at each point. Since $\mathrm{d}\mathbf{r}$ is a vector tangent to the surface, it must decompose into tangent vectors

$$\mathrm{d}\mathbf{r} = \boldsymbol{\mathcal{A}}_1\mathbf{e}_1 + \boldsymbol{\mathcal{A}}_2\mathbf{e}_2, \tag{2.1}$$

where $\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2$ are 1-forms. Equation (2.1) is the first structure equation of the surface. The second one consists of differentials of frame $\mathbf{e}$,

$$\mathrm{d}\mathbf{e} + \mathbf{\Omega}\mathbf{e} = 0, \tag{2.2}$$

where $\mathbf{\Omega}$ is an antisymmetric matrix of 1-forms. The integrability conditions of (2.1) and (2.2) reduce to

$$\mathrm{d}\boldsymbol{\mathcal{A}} + \mathbf{\Omega} \wedge \boldsymbol{\mathcal{A}} = 0, \qquad \boldsymbol{\mathcal{A}} = (\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2, 0)^T, \tag{2.3}$$

$$\mathrm{d}\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = 0. \tag{2.4}$$

Writing matrix $\mathbf{\Omega}$ in the following way

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\omega_{12} & -\omega_{13} \\ \omega_{12} & 0 & -\omega_{23} \\ \omega_{13} & \omega_{23} & 0 \end{pmatrix} \tag{2.5}$$

allows us to rewrite the structure equations (2.1)–(2.2) and the integrability conditions (2.4):

Structureequations	Integrabilityconditions	
$\mathrm{d}\mathbf{r} = \boldsymbol{\mathcal{A}}_1\mathbf{e}_1 + \boldsymbol{\mathcal{A}}_2\mathbf{e}_2,$	$\mathrm{d}\boldsymbol{\mathcal{A}}_1 = \omega_{12} \wedge \boldsymbol{\mathcal{A}}_2,$	(2.6)
$\mathrm{d}\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3,$	$\mathrm{d}\boldsymbol{\mathcal{A}}_2 = -\omega_{12} \wedge \boldsymbol{\mathcal{A}}_1,$	(2.7)
$\mathrm{d}\mathbf{e}_2 = -\omega_{12}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3,$	$\omega_{13} \wedge \boldsymbol{\mathcal{A}}_1 + \omega_{23} \wedge \boldsymbol{\mathcal{A}}_2 = 0,$	(2.8)
$\mathrm{d}\mathbf{e}_3 = -\omega_{13}\mathbf{e}_1 - \omega_{23}\mathbf{e}_2,$	$\mathrm{d}\omega_{12} = \omega_{23} \wedge \omega_{13},$	(2.9)
	$\mathrm{d}\omega_{13} = \omega_{12} \wedge \omega_{23},$	(2.10)
	$\mathrm{d}\omega_{23} = \omega_{13} \wedge \omega_{12}.$	(2.11)

The first and second fundamental forms induced on the surface are given by

$$g_I = \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = \boldsymbol{\mathcal{A}}_1 \otimes \boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{A}}_2 \otimes \boldsymbol{\mathcal{A}}_2, \tag{2.12}$$

$$g_{II} = -\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{N} = \boldsymbol{\mathcal{A}}_1 \otimes \omega_{13} + \boldsymbol{\mathcal{A}}_2 \otimes \omega_{23}, \tag{2.13}$$

where dot $\cdot$ denotes a scalar product in $\mathbb{E}^3$. Since the normal vector defines a map $\mathbf{r} \mapsto \mathbf{N}$ on $\mathbf{\Sigma}$ into a two-dimensional sphere S_2, the following form

$$g_{III} = \mathrm{d}\mathbf{N} \cdot \mathrm{d}\mathbf{N} = \boldsymbol{\omega}_{13} \otimes \boldsymbol{\omega}_{13} + \boldsymbol{\omega}_{23} \otimes \boldsymbol{\omega}_{23} \tag{2.14}$$

is a metric on the sphere, called the third fundamental form of the surface.

Aside from umbilical points the second fundamental form has two distinct eigenvalues: κ_1 and κ_2 with respect to g_I at every point. The functions κ_1 and κ_2 are called the principal curvatures of $\mathbf{\Sigma}$. Therefore,

$$\boldsymbol{\omega}_{13} = \kappa_1 \boldsymbol{\mathcal{A}}_1, \qquad \boldsymbol{\omega}_{23} = \kappa_2 \boldsymbol{\mathcal{A}}_2 \tag{2.15}$$

and the second and third fundamental forms can be expressed as

$$g_{II} = \kappa_1 \boldsymbol{\mathcal{A}}_1 \otimes \boldsymbol{\mathcal{A}}_1 + \kappa_2 \boldsymbol{\mathcal{A}}_2 \otimes \boldsymbol{\mathcal{A}}_2, \tag{2.16}$$

$$g_{III} = \kappa_1^2 \boldsymbol{\mathcal{A}}_1 \otimes \boldsymbol{\mathcal{A}}_1 + \kappa_2^2 \boldsymbol{\mathcal{A}}_2 \otimes \boldsymbol{\mathcal{A}}_2. \tag{2.17}$$

Choosing the local coordinates (α, β) there exist two functions $A_1 = A_1(\alpha, \beta)$ and $A_2 = A_2(\alpha, \beta)$ such that

$$\boldsymbol{\mathcal{A}}_1 = A_1 \,\mathrm{d}\alpha, \qquad \boldsymbol{\mathcal{A}}_2 = A_2 \,\mathrm{d}\beta. \tag{2.18}$$

Consequently,

$$\boldsymbol{\omega}_{12} = -\frac{A_{1\beta}}{A_2}\,\mathrm{d}\alpha + \frac{A_{2\alpha}}{A_1}\,\mathrm{d}\beta \tag{2.19}$$

and the integrability conditions (2.6)–(2.8) are automatically satisfied, while (2.9)–(2.11) reduce to the Gauss–Mainardi–Codazzi equations

$$A_1 A_2 \kappa_1 \kappa_2 + \left(\frac{A_{2\alpha}}{A_1}\right)_\alpha + \left(\frac{A_{1\beta}}{A_2}\right)_\beta = 0, \tag{2.20}$$

$$\kappa_{2\alpha} + \frac{A_{2\alpha}}{A_2}(\kappa_2 - \kappa_1) = 0, \tag{2.21}$$

$$\kappa_{1\beta} + \frac{A_{1\beta}}{A_1}(\kappa_1 - \kappa_2) = 0, \tag{2.22}$$

where α and β in subscripts denote partial derivatives. Now the three forms (2.12)–(2.14) are

$$g_I = A_1^2 \,\mathrm{d}\alpha^2 + A_2^2 \,\mathrm{d}\beta^2, \tag{2.23}$$

$$g_{II} = A_1^2 \kappa_1 \,\mathrm{d}\alpha^2 + A_2^2 \kappa_2 \,\mathrm{d}\beta^2, \tag{2.24}$$

$$g_{III} = A_1^2 \kappa_1^2 \,\mathrm{d}\alpha^2 + A_2^2 \kappa_2^2 \,\mathrm{d}\beta^2. \tag{2.25}$$

The first and second fundamental forms g_I, g_{II} are purely diagonal and (α, β) are said to constitute the curvature coordinates. It is readily seen that the third fundamental form can be expressed in terms of g_I and g_{II}:

$$g_{III} = 2\mathcal{H} g_{II} - \mathcal{K} g_I, \tag{2.26}$$

where

$$\mathcal{H} = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad \mathcal{K} = \kappa_1 \kappa_2 \tag{2.27}$$

are the mean and Gauss curvatures of the surface $\mathbf{\Sigma}$.

3. L-isothermic surfaces

The assumption that a surface is L-isothermic (has an isothermic representation of its lines of curvature) is equivalent to the following condition:

$$\left(\log\left|\frac{A_1\kappa_1}{A_2\kappa_2}\right|\right)_{\alpha\beta} = 0, \tag{3.1}$$

which by suitable reparametrization of the lines of curvature may be replaced by $A_1\kappa_1 = \pm A_2\kappa_2$. Here we proceed further with the following constraint:

$$A_1\kappa_1 = A_2\kappa_2. \tag{3.2}$$

The geometry of two-dimensional surfaces for which condition (3.2) is satisfied can be described by an ordinary nonhomogeneous second-order differential equation with nonhomogeneity given by solution of another equation. The derivation of the equations was described in [19]. Here, we recall it using the same notation. It is advantageous to introduce a new function θ,

$$\mathrm{e}^{\theta} = -A_1\kappa_1 = -A_2\kappa_2 \tag{3.3}$$

and reduce the Gauss-Mainardi-Codazzi equations (2.20)–(2.22) to the following form:

$$A_{2\alpha} = A_1\theta_\alpha, \quad A_{1\beta} = A_2\theta_\beta, \tag{3.4}$$

$$\theta_{\alpha\alpha} + \theta_{\beta\beta} + \mathrm{e}^{2\theta} = 0. \tag{3.5}$$

It is observed that equations (3.4) are invariant with respect to the following Lie-point symmetries

$$\begin{aligned} A_1 &\mapsto A_1' = A_1 + c_1\,\mathrm{e}^{\theta} + c_2\,\mathrm{e}^{-\theta}, \\ A_2 &\mapsto A_2' = A_2 + c_1\,\mathrm{e}^{\theta} - c_2\,\mathrm{e}^{-\theta}, \end{aligned} \tag{3.6}$$

where $c_1, c_2 \in \mathbb{R}$. The geometric interpretation of transformations (3.6) will be given in the following section.

When (3.3) is substituted in (2.25) the latter is reducible to

$$g_{III} = \mathrm{e}^{2\theta}(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2), \tag{3.7}$$

so that the third fundamental form is conformally flat in the coordinate system (α, β). The general solution of the Liouville equation (3.5) is given by

$$\mathrm{e}^{\theta} = \frac{2|\rho_z|}{1+\rho\bar{\rho}}, \tag{3.8}$$

where $\rho = \rho(z)$ is an arbitrary holomorphic function of complex coordinate $z = \alpha + \mathrm{i}\beta$. Rewriting ρ using the two holomorphic functions $\Phi_1(z) \neq 0$ and $\Phi_2(z)$,

$$\rho = \frac{\Phi_2}{\Phi_1}, \tag{3.9}$$

subject to the Wronskian condition

$$\Phi_1\Phi_{2z} - \Phi_{1z}\Phi_2 = 1 \tag{3.10}$$

we find that (3.8) may be written in the form

$$\mathrm{e}^{-\theta} = \tfrac{1}{2}(|\Phi_1|^2 + |\Phi_2|^2). \tag{3.11}$$

If we define a holomorphic function $\mathsf{U}(z)$ according to

$$\mathsf{U} = -\frac{\Phi_{1zz}}{\Phi_1} \tag{3.12}$$

then Φ_1 is a solution of the linear ordinary second-order differential equation

$$\Phi_{zz} + \mathsf{U}\Phi = 0 \tag{3.13}$$

with potential U. On use of (3.10), it can be shown that the function Φ_2 is the second solution of (3.13), linearly independent of Φ_1. Moreover the potential U can be calculated from ρ

$$\mathsf{U} = \tfrac{1}{2}\{\rho, z\}, \tag{3.14}$$

where $\{\cdot, \cdot\}$ is a Schwarzian derivative defined by

$$\{\rho, z\} = \left(\frac{\rho_{zz}}{\rho_z}\right)_z - \frac{1}{2}\left(\frac{\rho_{zz}}{\rho_z}\right)^2.$$

Therefore, the general solution (3.8) of the Liouville equation (3.5) may be expressed in terms of two holomorphic solutions $\Phi_1 \neq 0$ and Φ_2 of equation (3.13) subject to the Wronskian condition (3.10) with an arbitrary potential $\mathsf{U}(z)$.

To find the position vector $\mathbf{r}$ to the surface $\mathbf{\Sigma}$, we exploit the tangential coordinate

$$b = \mathbf{r} \cdot \mathbf{N}, \tag{3.15}$$

which is the distance of the tangent plane to $\mathbf{\Sigma}$ from the origin. The following proposition has been given in [19].

Proposition 1. *Let $T_0 = T_0(z, \bar{z})$ be a real particular solution of the complex inhomogeneous equation*

$$T_{zz} + \mathsf{U}T = \frac{P}{4}, \tag{3.16}$$

where $\mathsf{U} = \mathsf{U}(z)$ is an arbitrary holomorphic function of $z = \alpha + \mathrm{i}\beta$ and $P = P(z, \bar{z})$ is a real solution of the Moutard equation

$$P_{\alpha\beta} = 2P\,\mathrm{Im}\,\mathsf{U}. \tag{3.17}$$

If Φ_1 and Φ_2 are two linearly independent solutions of homogeneous version of (3.16) then the position vector $\mathbf{r}$ of a class of parallel surfaces in $\mathbb{E}^3$ with isothermic representation of its lines of curvature (L-minimal surfaces) adopts the form

$$\mathbf{r} = \mathrm{e}^{-\theta} b_z \mathbf{I} + \mathrm{e}^{-\theta} b_{\bar{z}} \bar{\mathbf{I}} + b\mathbf{N}, \tag{3.18}$$

where the tangential coordinate b is given by

$$b = \mathrm{e}^{\theta} T_0 + \mathfrak{b}, \qquad \mathfrak{b} \in \mathbb{R} \tag{3.19}$$

and θ is defined by (3.11). The unit tangent vectors $\mathbf{X}, \mathbf{Y}$ and the normal vector $\mathbf{N}$ are determined by

$$\begin{aligned}
\mathbf{I} = \mathbf{X} + \mathrm{i}\mathbf{Y} &= \frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix} \Phi_2^2 - \Phi_1^2 \\ \mathrm{i}\left(\Phi_1^2 + \Phi_2^2\right) \\ 2\Phi_1\Phi_2 \end{pmatrix}, \\
\mathbf{N} &= -\frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix} \Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2 \\ \mathrm{i}(\bar{\Phi}_1\Phi_2 - \Phi_1\bar{\Phi}_2) \\ |\Phi_1|^2 - |\Phi_2|^2 \end{pmatrix}.
\end{aligned} \tag{3.20}$$

The coefficients of the first fundamental form (2.23) can be calculated from

$$P = A_1 - A_2, \tag{3.21}$$

$$R = A_1 + A_2, \tag{3.22}$$

where

$$R = 4\,\mathrm{e}^{-\theta} b_{z\bar{z}} + 2\,\mathrm{e}^{\theta} b. \tag{3.23}$$

It should be noted that the real solution to equation (3.16) always exists. In fact equation (3.17) is a necessary and sufficient condition for existence of a real solution to (3.16). The derivation of formulae (3.20) for **I** and **N** has been described in [19]. It should be remarked that the potential U in equation (3.17) is always defined up to addition of a real constant. It can be shown that this constant corresponds to a parameter in the transformation of an orthogonal net with an isothermic spherical representation considered by Bianchi and Eisenhart [3].

4. Laguerre transformation

We observe that if T_0 is a real solution of (3.16), so also is

$$T_0' = T_0 + a_1|\Phi_1|^2 + a_2\Phi_1\bar{\Phi}_2 + \bar{a}_2\bar{\Phi}_1\Phi_2 + a_3|\Phi_2|^2, \tag{4.1}$$

where a_1, a_3 are real constants and a_2 is a complex constant. There is also a freedom in Φ_1 and Φ_2. In fact, any linear combination of Φ_1, Φ_2

$$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \mapsto \begin{pmatrix}\Phi_1'\\ \Phi_2'\end{pmatrix} = \mathbf{S}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} s_1 & s_2\\ s_3 & s_4\end{pmatrix} \in SL(2,\mathbb{C}) \tag{4.2}$$

gives rise to another surface. The full interpretation of transformations (4.1) and (4.2) will be given in this section. According to proposition 1, any real particular solution of (3.16) determines the tex-parameter class of surfaces defined by

$$b' = \frac{2(T_0 + a_1|\Phi_1|^2 + a_2\Phi_1\bar{\Phi}_2 + \bar{a}_2\bar{\Phi}_1\Phi_2 + a_3|\Phi_2|^2)}{|\Phi_1|^2 + |\Phi_2|^2}, \tag{4.3}$$

where Φ_1, Φ_2 can undergo the linear transformation (4.2). It will be shown that the constants a_i $(i = 1, 2, 3)$ do not change the class of parallel surfaces. Hence b can be always defined by (3.19), as it stands in proposition 1. For any ten real parameters: a_i $(i = 1, 2, 3)$ and s_j $(j = 1, 2, 3, 4)$ the third fundamental form (3.7) is conformally flat in (α, β), so we may expect that these parameters correspond to ten-dimensional Laguerre group.

We recall the definition of Laguerre transformation [1, 2, 5]. Consider a six-dimensional space $\mathbb{R}^6$ with the scalar product h of signature $(-++++-)$ and the Lie quadric $\mathcal{L}$ defined by

$$\mathbb{R}^6 \supset \mathcal{L} = \{v \in \mathbb{R}^6 | h(v, v) = 0\}. \tag{4.4}$$

We can associate a sphere $Sph(\mathbf{p}, \mathcal{R})$ in $\mathbb{E}^3$ of radius $\mathcal{R}$ and center $\mathbf{p}$ with any point of $\mathcal{L}$

$$Sph(\mathbf{p}, \mathcal{R}) \quad \leftrightsquigarrow \quad \left(\frac{1 + \mathbf{p}^2 - \mathcal{R}^2}{2}, \frac{1 - \mathbf{p}^2 + \mathcal{R}^2}{2}, \mathbf{p}, \mathcal{R}\right) \in \mathcal{L}. \tag{4.5}$$

A pair of two-dimensional submanifolds

$$\xi = \left(\frac{1 + \mathbf{r}^2 + 2\rho_1 b}{2}, \frac{1 - \mathbf{r}^2 - 2\rho_1 b}{2}, \mathbf{r} + \rho_1\mathbf{N}, \rho_1\right), \tag{4.6}$$

$$\boldsymbol{\eta} = \left(\frac{1 + \mathbf{r}^2 + 2\rho_2 b}{2}, \frac{1 - \mathbf{r}^2 - 2\rho_2 b}{2}, \mathbf{r} + \rho_2 \mathbf{N}, \rho_2 \right) \tag{4.7}$$

of $\mathcal{L}$ corresponds to two-dimensional surface $\mathbf{r}$ with curvature spheres: $Sph(\mathbf{r} + \rho_1 \mathbf{N}, \rho_1)$, $Sph(\mathbf{r}+\rho_2\mathbf{N}, \rho_2)$, where $\rho_1 = \frac{1}{\kappa_1}$ and $\rho_2 = \frac{1}{\kappa_2}$ denote the radii of principal curvature and b is defined in (3.15). The Laguerre group is a subgroup of $SO(2, 4)$ acting in $\mathbb{R}^6$ and it is isomorphic to ten-dimensional group $\mathbb{R}^4 \rtimes SO(1, 3)$. Therefore, there exists a correspondence between the parameters a_i, s_j $(i = 1, 2, 3, j = 1, 2, 3, 4)$ and the ten parameters of the Poincaré Lie algebra $\mathfrak{g} = \mathbb{R}^4 \ni\!\!\!\rhd so(1, 3)$. The lie algebra $\mathfrak{g}$ naturally occurs in special theory of relativity and thus we use terms which come from it (translation, boost, dilation).

Let $\mathbf{r}$ be a position vector (3.18) of a surface defined by T_0 (a real solution of inhomogeneous equation (3.16)) and functions Φ_1, Φ_2. The transformations of $\mathbf{r}$ can be determined as follows.

(i) *Translation in space (the transformation generated by the Abelian three-dimensional subalgebra $\mathbb{R}^3 \subset \mathfrak{g}$)*

The transformation

$$\begin{cases} T_0 & \mapsto & T_0' = T_0 + a_1(|\Phi_1|^2 - |\Phi_2|^2) + a_2\Phi_1\bar{\Phi}_2 + \bar{a}_2\bar{\Phi}_1\Phi_2 \\ \Phi_1 & \mapsto & \Phi_1' = \Phi_1 \\ \Phi_2 & \mapsto & \Phi_2' = \Phi_2 \end{cases} \tag{4.8}$$

corresponds to translation of the surface in $\mathbb{E}^3$

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} + \begin{pmatrix} -2\,\mathrm{Re}\, a_2 \\ 2\,\mathrm{Im}\, a_2 \\ -2a_1 \end{pmatrix}. \tag{4.9}$$

(ii) *Rotation in space (the transformation generated by the subalgebra $so(3) \subset \mathfrak{g}$)*

The transformation

$$\begin{cases} T_0 & \mapsto & T_0' = T_0 \\ \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} & \mapsto & \begin{pmatrix}\Phi_1' \\ \Phi_2'\end{pmatrix} = \mathbf{S_R}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}, \qquad \mathbf{S_R} \in SU(2) \subset SL(2, \mathbb{C}) \end{cases} \tag{4.10}$$

corresponds to rotation of the surface in $\mathbb{E}^3$

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{M}\mathbf{r}, \qquad \mathbf{M} \in SO(3). \tag{4.11}$$

(iii) *Dilation (the transformation generated by one-dimensional subalgebra $\mathbb{R}^1 \subset \mathfrak{g}$)*

The transformation

$$\begin{cases} T_0 & \mapsto & T_0' = T_0 + a_1(|\Phi_1|^2 + |\Phi_2|^2) = T_0 + 2a_1\,\mathrm{e}^{-\theta} \\ \Phi_1 & \mapsto & \Phi_1' = \Phi_1 \\ \Phi_2 & \mapsto & \Phi_2' = \Phi_2 \end{cases} \tag{4.12}$$

corresponds to the following transformation of a surface:

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} + 2a_1\mathbf{N}. \tag{4.13}$$

Therefore, (4.12) adds the constant $2a_1$ to the foliation parameter: $\mathfrak{b} \mapsto \mathfrak{b}' = \mathfrak{b} + 2a_1$. Note that this transformation does not alter P but changes $R \mapsto R' = R + 4a_1\,\mathrm{e}^{\theta}$ (cf (3.23)). The Lie-point symmetry (3.6) for $c_1 = 2a_1$ and $c_2 = 0$ corresponds to (4.13).

(iv) *Boost*

The transformation

$$\begin{cases} T_0 & \mapsto \quad T_0' = T_0 \\ \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} & \mapsto \quad \begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \mathbf{S_L} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \end{cases} \tag{4.14}$$

where

$$\mathbf{S_L} = \begin{pmatrix} \cosh\left(\frac{n}{2}\right) - \sinh\left(\frac{n}{2}\right) n_3 & -\sinh\left(\frac{n}{2}\right)(n_1 + \mathrm{i}n_2) \\ -\sinh\left(\frac{n}{2}\right)(n_1 - \mathrm{i}n_2) & \cosh\left(\frac{n}{2}\right) + \sinh\left(\frac{n}{2}\right) n_3 \end{pmatrix} \in SL(2, \mathbb{C}) \tag{4.15}$$

corresponds to the following transformation of a surface:

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} - \frac{\sinh(n)\mathbf{N} + (\cosh(n) - 1)\mathbf{n}}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}} \mathbf{r}\cdot\mathbf{n}, \tag{4.16}$$

where $n \in \mathbb{R}$ and $\mathbf{n} = (n_1, n_2, n_3)$ is a constant real unit vector. Here, we set $\mathfrak{b} = 0$ in definition (3.19) of b[1]. The transformation of the tangent and normal vectors of the surface can be found in appendix A.

5. Weierstrass representation of L-isothermic surfaces which are L-minimal

The two simplest examples: (i) a sphere and (ii) a minimal surface correspond to the following solutions of Mainardi–Codazzi equations (3.4):

(i) $A_1 = A_2 = \mathrm{e}^{\theta} \quad \Rightarrow \quad P = 0,$

(ii) $A_1 = -A_2 = \mathrm{e}^{-\theta} \quad \Rightarrow \quad P = 2\mathrm{e}^{-\theta}.$

In the former case (i) the particular solution $T_0 = 0$ ($b = \mathfrak{b} =$ const) and the position vector of the sphere reads

$$\mathbf{r} = \mathfrak{b}\mathbf{N}. \tag{5.1}$$

In the latter case the position vector of the class of surfaces parallel to the minimal surface adopts the form

$$\mathbf{r} = \mathrm{Re} \begin{pmatrix} \int \left(\Phi_2^2 - \Phi_1^2\right) \mathrm{d}z \\ \mathrm{i} \int \left(\Phi_1^2 + \Phi_2^2\right) \mathrm{d}z \\ 2 \int \Phi_1 \Phi_2 \, \mathrm{d}z \end{pmatrix} + \mathfrak{b}\mathbf{N}. \tag{5.2}$$

The minimal surface is given by (5.2) with $\mathfrak{b} = 0$. This result allows one to interpret the Lie-point symmetry (3.6) for $c_1 = 0$. If $\mathbf{r}_0$ is the position vector of a surface $\mathbf{\Sigma}_0$ defined by functions P, Φ_1 and Φ_2 then the following transformation

$$P \mapsto P' = P + 2c_2\,\mathrm{e}^{-\theta}, \qquad R \mapsto R' = R \tag{5.3}$$

corresponds to the transformation of the surface

$$\mathbf{r}_0 \mapsto \mathbf{r}' = \mathbf{r}_0 + \mathbf{r}_{\min}, \tag{5.4}$$

where

$$\mathbf{r}_{\min} = \frac{c_2}{2} \mathrm{Re} \begin{pmatrix} \int \left(\Phi_2^2 - \Phi_1^2\right) \mathrm{d}z \\ \mathrm{i} \int \left(\Phi_1^2 + \Phi_2^2\right) \mathrm{d}z \\ 2 \int \Phi_1 \Phi_2 \, \mathrm{d}z \end{pmatrix} \tag{5.5}$$

is a minimal surface with tangent planes parallel to tangent planes to the surface $\mathbf{\Sigma}_0$.

[1] Note that for any given surface, $\mathfrak{b}$ can be always set to zero by suitable redefinition of T_0 in (3.19).

From equation (3.17) it is seen that an additional constraint on P may imply a constraint on potential U. If $\left(P^n\right)_{\alpha\beta} = 0$ then $\mathsf{U} = \frac{1}{n}\left(1 - \frac{1}{n}\right)\wp(z) + \text{const}$, where $\wp(z)$ is the Weierstrass elliptic function and (3.13) reduces to the Lamé equation. The case $n = 2$ has been described in [19], while $n = 1$ corresponds to surfaces with plane lines of curvature [4, 10]. If P depends on one variable only the surfaces are necessarily canal surfaces (they were called *generalized Dupin cyclides* in [20]). The condition $P = \text{const}$ is the necessary and sufficient condition for a surface to be a Dupin cyclide [19].

The L-minimal surfaces are analogs of the minimal surfaces in Euclidean space. They are defined as surfaces which satisfy the fourth-order differential equation

$$\Delta_{III}\left(\frac{\mathcal{H}}{\mathcal{K}}\right) = 0, \tag{5.6}$$

where Δ_{III} is a Laplace operator with respect to the third fundamental form and $\mathcal{H}$, $\mathcal{K}$ are the mean and Gauss curvatures. The method described in proposition 1 allows one to find the position vector of L-isothermic surfaces which are L-minimal.

Proposition 2. *The L-isothermic surfaces, associated with potential* U*, are L-minimal if* P *satisfies*

$$P_{zz} + \mathsf{U}P = 0. \tag{5.7}$$

Proof. According to (3.7) the Laplace operator Δ_{III} is given by $\Delta_{III} = 4\mathrm{e}^{-2\theta}\partial_{z\bar{z}}$ and the ratio $\mathcal{H}/\mathcal{K}$ can be rewritten in the following way (cf (3.3)):

$$\frac{\mathcal{H}}{\mathcal{K}} = -\frac{1}{2}R\,\mathrm{e}^{-\theta}. \tag{5.8}$$

By virtue of Mainardi equations: $(\mathrm{e}^{-\theta}R)_{\bar{z}} = \mathrm{e}^{-2\theta}(\mathrm{e}^{\theta}P)_z$, it is readily seen that (5.6) can be reduced to

$$P_{zz} - \mathrm{e}^{\theta}(\mathrm{e}^{-\theta})_{zz}P = 0 \tag{5.9}$$

which, on use of (3.11) and (3.12), is equivalent to (5.7)[2]. □

According to proposition 2 the function P can be expressed in terms of Φ_1 and Φ_2 (solutions of (3.13))

$$P = m_1|\Phi_1|^2 + m_2\Phi_1\bar{\Phi}_2 + \bar{m}_2\bar{\Phi}_1\Phi_2 + m_3|\Phi_2|^2, \tag{5.10}$$

where $m_1, m_3 \in \mathbb{R}$ and $m_2 \in \mathbb{C}$. Using a linear transformation of Φ_1 and Φ_2 (which corresponds to the Laguerre transformation of a surface) the nonzero Hermitian form (5.10) can be always reduced to the following form:

$$P = |\Phi_1|^2 + \varepsilon|\Phi_2|^2, \tag{5.11}$$

where $\varepsilon = -1, 0, 1$. The multiplicative factor in P is omitted without loss of generality.

Proposition 3. *Any surface which is both L-isothermic and L-minimal can be mapped (by the Laguerre transformation) to one of the following three classes of surfaces:*

(1) Surfaces whose central sphere congruence has centers lying in the plane $z = 0$ *in* $\mathbb{E}^3$*:*

$$\mathfrak{L}_{\mathfrak{m}1}\colon\quad \mathbf{r_1} = \mathrm{Re}\begin{pmatrix} -\int(1+\rho^2)F(\rho)\,\mathrm{d}\rho \\ \mathrm{i}\int(1-\rho^2)F(\rho)\,\mathrm{d}\rho \\ 0 \end{pmatrix} - \big(2\,\mathrm{Re}\big(\textstyle\int\rho F(\rho)\,\mathrm{d}\rho\big) - \mathfrak{b}\big)\mathsf{N}, \tag{5.12}$$

[2] Note that the imaginary part of (5.7) is equivalent to (3.17), thus equation (5.7) does not imply any additional constraints on the potential U.

(2) Surfaces whose central sphere congruence is tangential to a fixed plane $z = \mathfrak{b}$ in $\mathbb{E}^3$:

$$\mathfrak{L}_{\mathfrak{m}2}:\quad \mathbf{r_2} = \operatorname{Re}\begin{pmatrix} -\int F(\rho)\,\mathrm{d}\rho \\ \mathrm{i}\int F(\rho)\,\mathrm{d}\rho \\ \int \rho F(\rho)\,\mathrm{d}\rho \end{pmatrix} - \left(\operatorname{Re}\left(\int \rho F(\rho)\,\mathrm{d}\rho\right) - \mathfrak{b}\right)\mathsf{N}, \tag{5.13}$$

(3) Surfaces parallel to minimal surfaces:

$$\mathbf{r_3} = \operatorname{Re}\begin{pmatrix} \int(\rho^2-1)F(\rho)\,\mathrm{d}\rho \\ \mathrm{i}\int(\rho^2+1)F(\rho)\,\mathrm{d}\rho \\ 2\int \rho F(\rho)\,\mathrm{d}\rho \end{pmatrix} + \mathfrak{b}\mathsf{N}, \tag{5.14}$$

where $F(\rho)$ is an arbitrary holomorphic function of ρ, and N is given by

$$\mathsf{N} = -\frac{1}{1+\rho\bar\rho}\begin{pmatrix} \rho+\bar\rho \\ \mathrm{i}(\rho-\bar\rho) \\ 1-\rho\bar\rho \end{pmatrix}. \tag{5.15}$$

Proof. The proof of this proposition is by construction. By the method of variation of parameters, the real particular solution of the complex inhomogeneous equation (3.16) with P given by (5.11) reads [21]

$$\begin{aligned} T_0 = \frac{1}{4}\Bigg(&\bar\Phi_1\Phi_2\int \Phi_1^2\,\mathrm{d}z - (|\Phi_1|^2-\varepsilon|\Phi_2|^2)\int \Phi_1\Phi_2\,\mathrm{d}z - \varepsilon\Phi_1\bar\Phi_2\int\Phi_2^2\,\mathrm{d}z \\ &+\Phi_1\bar\Phi_2\int\bar\Phi_1^2\,\mathrm{d}\bar z - (|\Phi_1|^2-\varepsilon|\Phi_2|^2)\int\bar\Phi_1\bar\Phi_2\,\mathrm{d}\bar z - \varepsilon\bar\Phi_1\Phi_2\int\bar\Phi_2^2\,\mathrm{d}\bar z\Bigg). \end{aligned} \tag{5.16}$$

According to proposition 1, the position vector adopts the form

$$\mathbf{r} = \frac{1}{2}\operatorname{Re}\begin{pmatrix} \int\left(-\Phi_1^2+\varepsilon\Phi_2^2\right)\mathrm{d}z \\ \mathrm{i}\int\left(\Phi_1^2+\varepsilon\Phi_2^2\right)\mathrm{d}z \\ (1+\varepsilon)\int\Phi_1\Phi_2\,\mathrm{d}z \end{pmatrix} + \left(\frac{\varepsilon-1}{2}\operatorname{Re}\left(\int\Phi_1\Phi_2\,\mathrm{d}z\right)+\mathfrak{b}\right)\mathbf{N} \tag{5.17}$$

where $\mathbf{N}$ is defined in (3.20). It is now convenient to change the coordinate z by taking $\rho = \rho(z)$ defined by (3.9) with $\mathrm{d}z = \Phi_1^2\,\mathrm{d}\rho$. Thus, the integrals contained in (5.17) can be expressed in the form

$$\begin{aligned} &\int\Phi_1^2\,\mathrm{d}z = 2\int F(\rho)\,\mathrm{d}\rho, \qquad \int\Phi_1\Phi_2\,\mathrm{d}z = 2\int\rho F(\rho)\,\mathrm{d}\rho, \\ &\int\Phi_2^2\,\mathrm{d}z = 2\int\rho^2F(\rho)\,\mathrm{d}\rho, \end{aligned} \tag{5.18}$$

where

$$F(\rho) = \tfrac{1}{2}\Phi_1^4. \tag{5.19}$$

The formulae (5.12)–(5.14) are obtained by substituting (5.18) into (5.17) and specifying $\varepsilon = -1, 0, 1$. □

The classification of L-isothermic surfaces which are also L-minimal presented in proposition 3 was already known to Blaschke [1] (see also [7]). The first, second and third fundamental forms of the L-minimal surfaces (5.12)–(5.14) are given in appendix B. It is readily verified that these surfaces are related by the linear condition:

$$\mathbf{r_3} = 2\mathbf{r_2} - \mathbf{r_1}. \tag{5.20}$$

It is remarked that some known surfaces may be retrieved from (5.12) or (5.13) by choosing proper functions $F(\rho)$ (for instance, putting $F(\rho) = \frac{1}{2}\,\mathrm{e}^{2\rho}$ in (5.13) the surface (6.36) from [19] is obtained. The same surface appears also in [16] (cf (40) therein)).

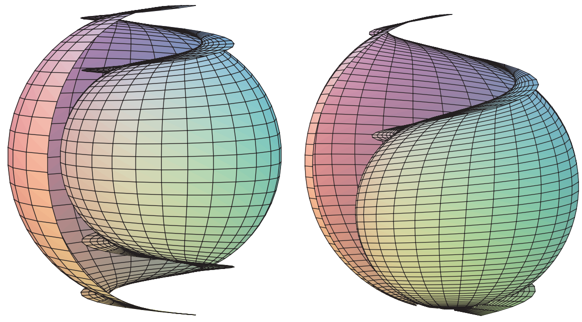

Figure 1. L-minimal helicoids: $\mathfrak{L}_{\mathfrak{m}1}$ (left) and $\mathfrak{L}_{\mathfrak{m}2}$ (right).

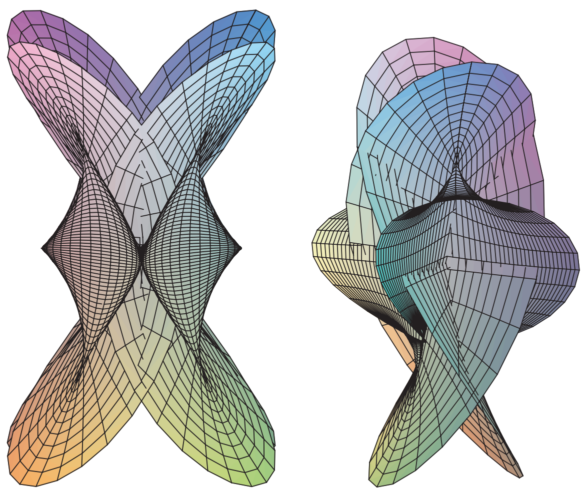

Figure 2. L-minimal surfaces of Henneberg: $\mathfrak{L}_{\mathfrak{m}1}$ (left) and $\mathfrak{L}_{\mathfrak{m}2}$ (right).

6. Examples: L-minimal helicoids and L-minimal surfaces of Henneberg

The helicoid is the minimal surface with Weierstrass representation associated with $F(\rho) = \frac{\mathrm{i}}{\rho^2}$. Two L-minimal surfaces $\mathfrak{L}_{\mathfrak{m}1}$, $\mathfrak{L}_{\mathfrak{m}2}$ related to helicoid can be parametrized in the following way:

$$\mathfrak{L}_{\mathfrak{m}1} - \text{helicoid:} \quad \mathbf{r_1}(u, v) = \begin{pmatrix} -2\cosh u \sin v \\ 2\cosh u \cos v \\ 0 \end{pmatrix} + \frac{2v}{\cosh u} \begin{pmatrix} \cos v \\ \sin v \\ \sinh u \end{pmatrix}, \tag{6.1}$$

$$\mathfrak{L}_{\mathfrak{m}2} - \text{helicoid:} \quad \mathbf{r_2}(u, v) = \begin{pmatrix} -\mathrm{e}^u \sin v \\ \mathrm{e}^u \cos v \\ v \end{pmatrix} + \frac{v}{\cosh u} \begin{pmatrix} \cos v \\ \sin v \\ \sinh u \end{pmatrix}. \tag{6.2}$$

They are displayed in figure 1. The minimal surface of Henneberg corresponds to $F(\rho) = 1 - \frac{1}{\rho^4}$. The L-minimal surfaces associated with it are given by (see figure 2)

$\mathfrak{L}_{\mathrm{m1}}$ – Henneberg:

$$\mathbf{r}_1(u,v) = \frac{2}{3}\begin{pmatrix} -3\cosh u\cos v - \cosh 3u\cos 3v \\ -3\cosh u\sin v + \cosh 3u\sin 3v \\ 0 \end{pmatrix} - \frac{2\cosh 2u\cos 2v}{\cosh u}\begin{pmatrix} -\cos v \\ \sin v \\ \sinh u \end{pmatrix}, \tag{6.3}$$

$\mathfrak{L}_{\mathrm{m2}}$ – Henneberg:

$$\mathbf{r}_2(u,v) = \frac{1}{3}\begin{pmatrix} -3\mathrm{e}^{u}\cos v - \mathrm{e}^{-3u}\cos 3v \\ -3\mathrm{e}^{u}\sin v + \mathrm{e}^{-3u}\sin 3v \\ 3\cosh 2u\cos 2v \end{pmatrix} - \frac{\cosh 2u\cos 2v}{\cosh u}\begin{pmatrix} -\cos v \\ \sin v \\ \sinh u \end{pmatrix}. \tag{6.4}$$

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Appendix A

The mapping (4.16) induces the following transformation of the tangent, **I**, and normal, **N**, vectors of a surface:

$$\mathbf{I} \mapsto \mathbf{I}' = \mathbf{I} - \frac{\sinh(n)\mathbf{N} + (\cosh(n) - 1)\mathbf{n}}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}}\mathbf{I}\cdot\mathbf{n}, \tag{A.1}$$

$$\mathbf{N} \mapsto \mathbf{N}' = \frac{\mathbf{N} + \sinh(n)\mathbf{n} + (\cosh(n) - 1)(\mathbf{N}\cdot\mathbf{n})\mathbf{n}}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}}. \tag{A.2}$$

The transformations of the other geometric quantities are given by

$$b \mapsto b' = \frac{b}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}}, \tag{A.3}$$

$$A_1 \mapsto A_1' = A_1 - \frac{\mathrm{e}^{\theta}\sinh(n)\mathbf{r}\cdot\mathbf{n}}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}}, \tag{A.4}$$

$$A_2 \mapsto A_2' = A_2 - \frac{\mathrm{e}^{\theta}\sinh(n)\mathbf{r}\cdot\mathbf{n}}{\cosh(n) + \sinh(n)\mathbf{N}\cdot\mathbf{n}}. \tag{A.5}$$

Appendix B

The geometric quantities of the set of L-minimal surfaces (5.12)–(5.14) are given by ($\varepsilon = -1, 0, 1$)

the first fundamental form:

$$\mathbf{r}_\rho\cdot\mathbf{r}_\rho = -\frac{2F(\rho)}{1+\rho\bar{\rho}}(1+\varepsilon\rho\bar{\rho})\frac{\mathcal{H}}{\mathcal{K}}, \tag{B.1}$$

$$\mathbf{r}_\rho\cdot\mathbf{r}_{\bar{\rho}} = \frac{F(\rho)\bar{F}(\bar{\rho})}{2}(1+\varepsilon\rho\bar{\rho})^2 + \frac{2\mathcal{H}^2}{\mathcal{K}^2(1+\rho\bar{\rho})^2}, \tag{B.2}$$

the second fundamental form:

$$\mathbf{r}_\rho \cdot \mathsf{N}_\rho = \frac{F(\rho)}{1+\rho\bar{\rho}}(1+\varepsilon\rho\bar{\rho}), \tag{B.3}$$

$$\mathbf{r}_\rho \cdot \mathsf{N}_{\bar{\rho}} = -\frac{2\mathcal{H}}{\mathcal{K}(1+\rho\bar{\rho})^2}, \tag{B.4}$$

the third fundamental form:

$$g_{III} = \frac{4\,\mathrm{d}\rho\,\mathrm{d}\bar{\rho}}{(1+\rho\bar{\rho})^2}, \tag{B.5}$$

where

$$\frac{\mathcal{H}}{\mathcal{K}} = (1-\varepsilon)\,\mathrm{Re}\int \rho F(\rho)\,\mathrm{d}\rho - \mathfrak{b}. \tag{B.6}$$

It is noted that the position vector of the set of surfaces (5.12)–(5.14) can also be represented in the form

$$\mathbf{r} = \mathbf{W} - \frac{\mathcal{H}}{\mathcal{K}}\mathsf{N}, \tag{B.7}$$

where

$$\mathbf{W} = \mathrm{Re}\begin{pmatrix} -\int(1-\varepsilon\rho^2)F(\rho)\,\mathrm{d}\rho \\ \mathrm{i}\int(1+\varepsilon\rho^2)F(\rho)\,\mathrm{d}\rho \\ (1+\varepsilon)\int\rho F(\rho)\,\mathrm{d}\rho \end{pmatrix}, \tag{B.8}$$

the ratio $\mathcal{H}/\mathcal{K}$ is given in (B.6) and N is defined in (5.15). For each point of the surface $\mathbf{r}$, $\mathbf{W}$ represents the sphere of radius $\frac{\mathcal{H}}{\mathcal{K}} = \frac{1}{2}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right)$ tangent to the surface $\mathbf{r}$ at this point.

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